Non-Homogeneous Eqns

1. Solve $y'' - 2y' - 3y = 3e^{2t}$

Soln:

Let $y = Ae^{2t}$

$(Ae^{2t})'' - 2(Ae^{2t})' - 3Ae^{2t} = 3e^{2t}$

$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t}$

$A = -1$

$y = -e^{2t}$ ← Particular Soln of N-H eqn

$y'' - 2y' - 3y' = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r_1 = 3, \; r_2 = -1$

$y = C_1e^{3t} + C_2e^{-t}$ ← General soln of the homogeneous eqn

$y = C_1e^{3t} + C_2e^{-t} - e^{2t}$ ← General soln of the N-H eqn

2. Solve $y'' + 2y' + y = 2e^{-t}$

Soln

Let $y = Ae^{-t}$

$(Ae^{-t})'' + 2(Ae^{-t})' + Ae^{-t} = 2e^{-t}$

$Ae^{-t} - 2Ae^{-t} + Ae^{-t} = 2e^{-t}$

$0 = 2e^{-t}$

Consider $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \rightarrow r = -1$

$y = C_1 e^{-t} + C_2 t e^{-t}$

This is a double resonance case.

Let $y = At^2 e^{-t}$

$(At^2e^{-t})'' + 2(At^2e^{-t})' + At^2e^{-t} = 2e^{-t}$

$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}}$
$\cancel{+At^2e^{-t}} = 2e^{-t}$

Recall that all terms with t or t^2 must go.

$2Ae^{-t} = 2e^{-t}$

$A = 1$

$y = t^2 e^{-t} \leftarrow$ Particular soln

$y = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}$

3. Solve $y'' + 4y = 3\sin(2t)$

Soln:

Let $y = A\cos(2t) + B\sin(2t)$

$(A\cos(2t) + B\sin(2t))'' + 4(A\cos(2t) + B\sin(2t))$
$= 3\sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) + 4A\cos(2t) + 4B\sin(2t)$
$= 3\sin(2t)$

$0 = 3\sin(2t)$

Consider $y'' + 4y = 0$

$y^2 + 4 = 0$

$y^2 = -4$

$y = \pm\sqrt{-4}$

$= \pm 2i$

Complex Resonance

$\lambda = 0, \; u = 2$

$y_1 = e^{\lambda t}\cos(ut)$ $\qquad$ $y_2 = e^{\lambda t}\sin(ut)$

$= \cos(2t)$ $\qquad$ $= \sin(2t)$

Let $y = Aty_1 + Bty_2$

$(Aty_1 + Bty_2)'' + 4(Aty_1 + Bty_2) = 3\sin(2t)$

$2Ay_1' + Aty_1'' + 2By_2' + Bty_2'' + 4Aty_1 +$

$4Bty_2 = 3\sin(2t)$

Sub in the y_1 and y_2 we found earlier.

$2A(\cos(2t))' + At(\cos(2t))'' + 2B(\sin(2t))' +$

$Bt(\sin(2t))'' + 4Bt\sin(2t) + 4At\cos(2t) = 3\sin(2t)$

$-4A\sin(2t) - \cancel{4At\cos(2t)} + 4B\cos(2t) +$

$(-\cancel{4Bt\sin(2t)}) + \cancel{4Bt\sin(2t)} + \cancel{4At\cos(2t)} = 3\sin(2t)$

$-4A\sin(2t) + 4B\cos(2t) = 3\sin(2t)$

$-4A = 3 \rightarrow A = -3/4$

$4B = 0 \rightarrow B = 0$

$y = -\dfrac{3t\cos(2t)}{4}$ $\leftarrow$ Particular soln

$$y = C_1\cos(2t) + C_2\sin(2t) - \frac{3t\cos(2t)}{4}$$